بِسمِ اللّٰهِ الرَّحمٰنِ الرَّحِيمِ

Devoir Surveillé $N°$ 7

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédation.

$\int$ y'

PCSI 1

Questions de Cours

Cours

Exercice 1

We have $1+\sin(x) = 1+x-\frac{x^3}{6}+o(x^3)$ and $\text{ch}(x) = 1+\frac{x^2}{2}+o(x^3)$, hence $(1+\sin(x))\,\text{ch}(x) = \left(1+x-\frac{x^3}{6}+o(x^3)\right)\left(1+\frac{x^2}{2}+o(x^3)\right) = 1+x+\frac{x^2}{2}+\frac{x^3}{2}+o(x^3)$

Exercice 2

$$\frac{\text{sh}(x)}{\sin(x)} = \frac{x+\frac{x^3}{6}+o(x^4)}{x-\frac{x^3}{6}+o(x^4)} = \frac{1+\frac{x^2}{6}+o(x^3)}{1-\frac{x^2}{6}+o(x^3)} = 1+\frac{x^2}{3}+o(x^3).$$

Exercice 3

Soit $P = X^5-1$.

1. z is a root of P if and only if $z^5-1=0$ which is equivalent to $z\in\mathbb{U}_5$. Hence $\{e^{\frac{2ik\pi}{5}}\ /\ k = 0,\dots,4\}$ is the set of roots of P.

2. Clearly $\deg P = 5$ and P has 5 roots and $\text{dom}(P) = 1$, hence $P = \prod_{k=0}^{4}(X-e^{\frac{2ik\pi}{5}})$.

3.

$$\begin{aligned} P &= (X-1)(X-e^{\frac{2i\pi}{5}})(X-e^{\frac{8i\pi}{5}})(X-e^{\frac{4i\pi}{5}})(X-e^{\frac{6i\pi}{5}}) \\ &= (X-1)(X-e^{\frac{2i\pi}{5}})(X-e^{-\frac{2i\pi}{5}})(X-e^{\frac{4i\pi}{5}})(X-e^{-\frac{4i\pi}{5}}) \\ &= (X-1)(X^2-2\cos(\frac{2\pi}{5})X+1)(X^2-2\cos(\frac{4\pi}{5})X+1) \end{aligned}$$

So $P = (X-1)\underbrace{(X^2-2\cos(\frac{2\pi}{5})X+1)}_{\Delta<0}\underbrace{(X^2-2\cos(\frac{4\pi}{5})X+1)}_{\Delta<0}$.

4. The poles of $\frac{1}{P}$ are all simples and $\deg \frac{1}{P} = -5 < 0$. Then the form of the decomposition to simples elements is : $\frac{1}{P} = \sum_{k=0}^{4} \frac{\alpha_k}{X - e^{\frac{2ik\pi}{5}}}$ Where $\alpha_k \in \mathbb{C}$.

Exercice 4

Soit $F = \frac{X}{(X-2)(X-1)^2} \in \mathbb{R}(X)$.

1. Clearly, $\frac{X}{(X-2)(X-1)^2}$ is an irreducible form of F. Hence F has one zero (0), and two poles 1 and 2.
2. $\deg F = -1 < 0$, hence its entire part is null. 1 is a double pole and 2 is simple pole. This justifies the following form :

$$F = \frac{\alpha}{X-1} + \frac{\beta}{(X-1)^2} + \frac{\gamma}{X-2}$$

Where $\alpha, \beta, \gamma \in \mathbb{R}$.

3. Calculus of α, β and γ :

We multiply by $(X-1)^2$, we get $\frac{X}{X-2} = \alpha(X-1) + \beta + \frac{\gamma(X-1)^2}{X-2}$. Now we apply at 1, we get $\beta = -1$.

We multiply by $(X-2)$, we get $\frac{X}{(X-1)^2} = \frac{\alpha(X-2)}{X-1} + \frac{\beta(X-2)}{(X-1)^2} + \gamma$. Now we apply at 2, we get $\gamma = 2$.

Finally, $0 = F(0) = -\alpha + \beta - \frac{\gamma}{2}$, hence $\alpha = -2$.

PROBLÈME

Étude d'une fonction et suite

Dans tout le problème, f désigne la fonction définie sur $]-\frac{\pi}{2}, \frac{\pi}{2}[$ par $f(x) = x + \tan(x)$.

Première partie : Étude de la fonction f

1. $f(-x) = -x + \tan(-x) = -x - \tan(x) = -f(x)$, so f is an odd function.
2. $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{6} + o(x^3)}{1 - \frac{x^2}{2} + o(x^3)} = (x - \frac{x^3}{6} + o(x^3))(1 + \frac{x^2}{2} + o(x^3)) = x + \frac{x^3}{3} + o(x^3)$. so $f(x) = 2x + \frac{x^3}{3} + o(x^3)$
3. Clearly, the function f is derivable and $f'(x) = 2 + \tan^2(x) > 0$. Hence f is strictly increasing. Moreover $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$ and $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\infty$.
4. Since f is increasing and $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$ and $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\infty$, we get $f(]-\frac{\pi}{2}, \frac{\pi}{2}[) =]-\infty, +\infty[= \mathbb{R}$.
5. f is strictly monotone, so it realize a bijection between $]-\frac{\pi}{2}, \frac{\pi}{2}[$ and its image $f(]-\frac{\pi}{2}, \frac{\pi}{2}[) = \mathbb{R}$

6. The function f is of the class $\mathscr{C}^\infty$, and $(\forall x \in]-\frac{\pi}{2},\frac{\pi}{2}[)\ f'(x) \neq 0$, so g is also of the class $\mathscr{C}^\infty$. In particular g has a $\mathrm{DL}_2(0)$. Now set $g(x) = a+bx+cx^2+o(x^2)$. Since $(g\circ f)(x) = x$, we get $a + b(2x) + c(2x)^2 + o(x^2) = x + o(x^2)$ that is $a + 2bx + 4cx^2 + o(x^2) = x + o(x^2)$. It follows that $a = 0$, $b = \frac{1}{2}$ and $c = 0$. Thus $g(x) = \frac{1}{2}x + o(x^2)$.

Deuxième partie :
Étude asymptotique d'une suite

On rappelle que g désigne la fonction f^{-1} définie dans la première partie du problème.
Pour $n \in \mathbb{N}$, on pose $u_n = g(n)$.

7. For $n \in \mathbb{N}$, we have $u_n = g(n) = f^{-1}(n)$, hence $f(u_n) = n$ that is $u_n + \tan(u_n) = n$.

8. For $n \in \mathbb{N}$, we have $u_n + \tan(u_n) = n$, so $u_n - n = \tan(u_n)$. Since $u_n \in]-\frac{\pi}{2},\frac{\pi}{2}[$, we get $u_n = \arctan(\tan(u_n)) = \arctan(u_n - n)$.

9. For $x \in \mathbb{R}_+^*$, set $h(x) = \arctan(x) + \arctan(\frac{1}{x})$. h is derivable on $\mathbb{R}_+^*$, and for any $x \in \mathbb{R}_+^*$, $f'(x) = 0$, so f is a constant function. Thus, for any $x \in \mathbb{R}_+^*$, $f(x) = f(1) = \frac{\pi}{2}$

10. $u_n = \arctan(n - u_n) = \frac{\pi}{2} - \arctan\left(\frac{1}{n-u_n}\right)$

11. Since $u_n \in]-\frac{\pi}{2},\frac{\pi}{2}[$, we have $n - \frac{\pi}{2} \leq n - u_n$, so $\lim\limits_{n\to+\infty}(n-u_n) = +\infty$. In particular $\arctan\left(\frac{1}{n-u_n}\right) \sim \frac{1}{n-u_n}$. On other hand $\dfrac{n-u_n}{n} = 1 - \dfrac{u_n}{n} \to 1$ that is $n - u_n \sim n$. Thus $\arctan\left(\frac{1}{n-u_n}\right) \sim \frac{1}{n-u_n} \sim \frac{1}{n}$, i.e $\arctan\left(\frac{1}{n-u_n}\right) = \frac{1}{n} + o(\frac{1}{n})$.

12. $u_n = \frac{\pi}{2} - \arctan\left(\frac{1}{n-u_n}\right) = \frac{\pi}{2} - \frac{1}{n} + o(\frac{1}{n})$.

13.

13.1 We have $\arctan'(x) = \dfrac{1}{1+x^2} = 1 - x^2 + o(x^2)$, hence $\arctan(x) = \arctan(0) + x - \frac{x^3}{3} + o(x^3) = x - \frac{x^3}{3} + o(x^3)$.

13.2 We have

$$\frac{1}{n-u_n} = \frac{1}{n}\frac{1}{1-\frac{u_n}{n}} = \frac{1}{n}\left(1 + \frac{u_n}{n} + \frac{u_n^2}{n^2} + o(\frac{u_n^2}{n^2})\right) = \frac{1}{n} + \frac{u_n}{n^2} + \frac{u_n^2}{n^3} + o(\frac{u_n^2}{n^3})$$

And by the question 12, we have $u_n = \frac{\pi}{2} - \frac{1}{n} + o(\frac{1}{n})$, so

$$\frac{u_n}{n^2} = \frac{\pi}{2n^2} - \frac{1}{n^3} + o(\frac{1}{n^3})$$

On other hand, since $\dfrac{u_n}{n^3} \sim \dfrac{\pi}{2n^3}$, we get

$$\frac{u_n}{n^3} = \frac{\pi^2}{4n^3} + o(\frac{1}{n^3})$$

Also, it is easy to see that $o(\dfrac{u_n^2}{n^3}) = o(\dfrac{1}{n^3})$. Hence

$$\frac{1}{n-u_n} = \frac{1}{n} + \frac{\pi}{2n^2} - \frac{1}{n^3} + \frac{\pi^2}{4n^3} + o(\frac{1}{n^3}) = \frac{1}{n} + \frac{\pi}{2n^2} + \frac{\pi^2-4}{4n^3} + o(\frac{1}{n^3})$$

Now

$$\begin{aligned}\arctan\left(\frac{1}{n-u_n}\right) &= \left(\frac{1}{n}+\frac{\pi}{2n^2}+\frac{\pi^2-4}{4n^3}\right)-\frac{1}{3}\left(\frac{1}{n}+\frac{\pi}{2n^2}+\frac{\pi^2-4}{4n^3}\right)^3+o(\frac{1}{n^3})\\ &= \frac{1}{n}+\frac{\pi}{2n^2}+\frac{\pi^2-4}{4n^3}-\frac{1}{3n^3}+o(\frac{1}{n^3})\\ &= \frac{1}{n}+\frac{\pi}{2n^2}+\frac{3\pi^2-16}{4n^3}+o(\frac{1}{n^3})\end{aligned}$$

It follows that

$$u_n=\frac{\pi}{2}-\frac{1}{n}-\frac{\pi}{2n^2}-\frac{3\pi^2-16}{12n^3}+o(\frac{1}{n^3})$$

END