بِسمِ اللهِ الرَّحمٰنِ الرَّحِيمِ

Devoir Surveillé $N°$ 5

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédation.

PCSI

Questions de Cours

Cours

Exercice 1

Soit E l'espace vectoriel $E=\mathbb{R}^3$, et considère les deux parties :

$$F=\{(x,y,z)\in E \ / \ x+y+z=0\} \quad \text{et} \quad G=\text{Vcet}((1,1,1))$$

1. Each element of G has the form $\lambda(1,1,1)=(\lambda,\lambda,\lambda)$, where $\lambda\in\mathbb{R}$.
2. Let $u=(x,y,z)$, $v=(x',y',z')$ be two elements of F and $\lambda\in\mathbb{R}$. We have $u+\lambda v=(x+\lambda x', y+\lambda y', z+\lambda z')$, on the other hand $(x+\lambda x')+(y+\lambda y')+(z+\lambda z')=x+y+z+\lambda(x'+y'+z')=0$. Hence $u+\lambda v\in F$.
3. Determination of a basis of F :
Let $u=(x,y,z)$ be any element of F. We have $x+y+z=0$, then $x=-y-z$, so $u=(-y-z,y,z)=y(-1,1,0)+z(-1,0,1)$. Note that $(-1,1,0),(-1,0,1)\in F$, hence $((-1,1,0),(-1,0,1))$ is a generated family of F. Let $\alpha,\beta\in\mathbb{R}$ such that $\alpha(-1,1,0)+\beta(-1,0,1)=0$, then $(-\alpha-\beta,\alpha,\beta)=0$, hence $\alpha=\beta=0$. It follows that $((-1,1,0),(-1,0,1))$ is a basis of F.
4. Let $u\in F\cap G$. Since $u\in G$, there exists $\lambda\in\mathbb{R}$, such that $u=(\lambda,\lambda,\lambda)$. We have $\lambda+\lambda+\lambda=0$ by the fact that $u\in F$. It follows that $\lambda=0$, and $u=0$.
5. Soit $u=(x,y,z)\in\mathbb{R}^3$. On pose $s=x+y+z$.
   - 5.1 If $s=0$, then $u\in F$.
   - 5.2 We have easily $(x-\frac{s}{3},y-\frac{s}{3},z-\frac{s}{3})+\frac{s}{3}(1,1,1)=(x-\frac{s}{3}+\frac{s}{3},y-\frac{s}{3}+\frac{s}{3},z-\frac{s}{3}+\frac{s}{3})=(x,y,z)$.
   - 5.3 Let $u=(x,y,z)\in E$. By the result of the previous question we have $u=(x-\frac{s}{3},y-\frac{s}{3},z-\frac{s}{3})+\frac{s}{3}(1,1,1)$. Since $(x-\frac{s}{3}+y-\frac{s}{3}+z-\frac{s}{3}=x+y+z-s=s-s=0$, we have $(x-\frac{s}{3},y-\frac{s}{3},z-\frac{s}{3}\in F$, on other hand $\frac{s}{3}(1,1,1)$. It follows that $u=(x-\frac{s}{3},y-\frac{s}{3},z-\frac{s}{3})+\frac{s}{3}(1,1,1)\in F+G$. Thus $E=F+G=F\oplus G$.

Exercice 2

Soit $f:\mathbb{R}^4\to\mathbb{R}$ l'application définie par $f(x,y,z,t)=x-2y+z-2t$.

1. Let $u=(x,y,z,t), v=(x',y',z',t')\in\mathbb{R}^4$ and $\lambda\in\mathbb{R}$.

$$\begin{array}{rcl} f(u+\lambda v) & = & f(x+\lambda x', y+\lambda y', z+\lambda z', t+\lambda t') \\ & = & (x+\lambda x')-2(y+\lambda y')+(z+\lambda z')-2(t+\lambda t') \\ & = & x-2y+z-2t+\lambda(x'-2y'+z'-2t') \\ & = & f(u)+\lambda f(v) \end{array}$$

Hence f is linear.

2. On considère les vecteurs $v_1=(2,1,0,0)$, $v_2=(-1,0,1,0)$ et $v_3=(2,0,0,1)$.
Let $u=(x,y,z,t)\in\ker f$, then $x-2y+z-2t=0$, hence $x=2y-z+2t$, so $u=(2y-z+2t,y,z,t)=y(2,1,0,0)+z(-1,0,1,0)+t(2,0,0,1)=yv_1+zv_2+tv_3$. It is easy to see that $v_1,v_2,v_3\in\ker f$. Hence (v_1,v_2,v_3) is a generated family of $\ker f$. Let $a,b,c\in\mathbb{R}$ such that $av_1+bv_2+cv_3=0$, then we have $(2a-b+2c,a,b,c)=0$, hence $a=b=c=0$. It follows that (v_1,v_2,v_3) is free, so it is a basis of $\ker f$.

3. Let $u\in\ker f\cap\mathrm{Vect}(1,1,1,1)$. We have $u\in\mathrm{Vect}(1,1,1,1)$, then there exists $\alpha\in\mathbb{R}$ such that $u=\alpha(1,1,1,1)=(\alpha,\alpha,\alpha,\alpha)$. On the other hand we have $u\in F$, which implies that $\alpha-2\alpha+\alpha-2\alpha=0$, hence $\alpha=0$. It follows that $u=0$.

PROBLÈME

Endomorphisme nilpotent

Soit E un espace vectoriel et $f\in\mathscr{L}(E)$ un endomorphisme de E (application linéaire de E vers E).
On rappelle que $f^0=\mathrm{Id}_E$ et pour $n\geq 1$, $f^n=\underbrace{f\circ f\circ\ldots\circ f}_{n \text{ fois}}$.

On dit que f est un endomorphisme nilpotent s'il existe un entier naturel $q\in\mathbb{N}$ tel que $f^q=0$.

Première partie : Exemples d'endomorphismes nilpotents

1. Soit $f:\mathbb{R}^3\to\mathbb{R}^3$ l'application définie par $f(x,y,z)=(2y,3z,0)$.

1.1 Let $u=(x,y,z), v=(x',y',z')\in\mathbb{R}^3$ and $\lambda\in\mathbb{R}$. We have

$$\begin{array}{rcl} f(u+\lambda v) & = & f(x+\lambda x', y+\lambda y', z+\lambda z') \\ & = & (2(y+\lambda y'), 3(z+\lambda z')) \\ & = & (2y,3z,0)+\lambda(2y',3z',0) \\ & = & f(u)+\lambda f(v) \end{array}$$

1.2 $f^2(x,y,z)=f(f(x,y,z))=f(2y,3z,0)=(2(3z),3\times 0,0)=(6z,0,0)$.

1.3 $f^3(x,y,z)=f(f^2(x,y,z))=f(3z,0,0)=(0,0,0)$, hence $f^3=0$, so f is nilpotent.

2. Soit $f:\mathbb{K}_n[X]\to\mathbb{K}_n[X]$ l'application définie par $f(P)=P'$.

2.1 Let $P,Q\in\mathbb{K}_n[X]$ and let $\lambda\in\mathbb{K}$. We have $f(P+\lambda Q)=(P+\lambda Q)'=P'+\lambda Q'=f(P)+\lambda f(Q)$. Hence f is linear.

2.2 Let $P\in\mathbb{K}_n[X]$. Since $\deg P\leq n$, we have $f^{n+1}(P)=P^{(n+1)}=0$, hence $f^{n+1}=0$. Thus f is nilpotent.

Deuxième partie : Indice de nilpotence

Dans cette partie f est un endomorphisme *nilpotent* de E.

3. Soit $A=\{k\in\mathbb{N}/\ f^k=0\}$.

3.1 Since f is nilpotent, there exists $q\in\mathbb{N}$ such that $f^q=0$. It follows that $q\in A$, so A is not empty.

3.2 A is a non empty subset of $\mathbb{N}$, hence it has a smallest element, say p.

3.3 We have $p\in A$, hence $f^p=0$. On other hand $p-1\notin A$ (since p is the smallest element of A), which implies $f^{p-1}\neq 0$.

4.

$$\begin{aligned}(\mathrm{Id}_E-f)\left(\sum_{k=0}^{p-1}f^k\right) &= \mathrm{Id}_E\left(\sum_{k=0}^{p-1}f^k\right)-f\left(\sum_{k=0}^{p-1}f^k\right)\\ &= \sum_{k=0}^{p-1}f^k-\sum_{k=0}^{p-1}f^{k+1}\\ &= \sum_{k=0}^{p-1}f^k-\sum_{k=1}^{p}f^k\\ &= \mathrm{Id}_E+\sum_{k=1}^{p-1}f^k-\sum_{k=1}^{p-1}f^k-f^p\\ &= \mathrm{Id}_E\end{aligned}$$

5. Denote $g=\sum_{k=0}^{p-1}f^k$. By the previous question we have $(\mathrm{Id}_E-f)g=\mathrm{Id}_E$. By the same argument one can show that $g(\mathrm{Id}_E-f)=\mathrm{Id}_E$, that is $(\mathrm{Id}_E-f)\circ g=\mathrm{Id}_E$ and $g\circ(\mathrm{Id}_-f)=\mathrm{Id}_E$. Thus Id_E-f is an isomorphism and $(\mathrm{Id}_E-f)^{-1}=g=\sum_{k=0}^{p-1}f^k$.

Troisième partie :
Cas d'indice de nilpotence égale à 3

Dans cette partie f est un endomorphisme nilpotent de E d'indice de nilpotence égale à 3 c'est-à-dire $f^3=0$ et $f^2\neq 0$.

6. Let y be an element of $\operatorname{Im} f^2$, then there exists $x\in E$ such that $y=f^2(x)$, so $f(y)=f(f^2(x))=f^3(x)=0$, hence $y\in\ker f$. It follows that $\operatorname{Im} f^2\subseteq\ker f$.

7. Clearly, since $f^2\neq 0$, there exists $v\in E$ such that $f^2(v)\neq 0$.

8. Let $a,b,c\in\mathbb{K}$, such that $av+bf(v)+cf^2(v)=0\ (\star)$. Applying f^2 to $(\star)$, we get $af^2(v)+bf^3(v)+cf^4(v)=0$, so $af^2(v)=0$, hence $a=0$ (since $f^2(v)\neq 0$). Now $(\star)$ become $bf(v)+cf^2(v)=0$. Applying f to $(\star)$ and this yields $bf^2(v)=0$, it follows that $b=0$. Finally we get $af(v)=0$ (from $(\star)$), so $af^2(v)=0$, hence $a=0$. Thus the family $(v,f(v),f^2(v))$ is free.

9. Let $a,b,c\in\mathbb{K}$ such that $a\mathrm{Id}_E+bf+cf^2=0$, that is, for all $x\in E$, $ax+bf(x)+cf^2(x)=0$. In particular $av+bf(v)+cf^2(v)=0$. Since the family $(v,f(v),f^2(v))$ is free, we have $a=b=c=0$.

10. If $x\in\ker f$, then $f(x)=0$, so $f^2(x)=f(f(x))=f(0)=0$. Hence $\ker f\subseteq\ker f^2$.

11. We have $f^2(f(v))=f^3(v)=0$, hence $f(v)\in\ker f^2$. On other hand $f(f(v))=f^2(v)\neq 0$, which implies that $f(v)\notin\ker f$. Thus $\ker f\neq\ker f^2$.

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