بِسمِ اللّٰهِ الرَّحمٰنِ الرَّحِيمِ

Devoir Surveillé $N°$ 7

dim — rg

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédation.

PCSI

Questions de Cours

Cours

Exercice 1

Soit A la matrice diagonale, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in \mathscr{M}_3(\mathbb{R})$. On note $\mathscr{C}(A) = \{M \in \mathscr{M}_3(\mathbb{R}) / \ AM = MA\}$

1. $0 \in \mathscr{C}(A)$ since $0.A = 0 = A.0$.
   Let $M, N \in \mathscr{C}(A)$ and $\lambda \in \mathbb{R}$. We have $(M+\lambda N)A = MA+\lambda NA = AM+\lambda AN = A(M+\lambda N)$, hence $M+\lambda N \in \mathscr{C}(A)$. It follows that $\mathscr{C}(A)$ is a vector subspace of $\mathscr{M}_3(\mathbb{R})$.

2. If $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is a diagonal matrix, then $AD = \begin{pmatrix} a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 3c \end{pmatrix} = DA$, hence $D \in \mathscr{C}(A)$.

3. By the previous question, we have $\{D \in \mathscr{M}_3(\mathbb{R}) / \ D \text{ diagonale } \} \subseteq \mathscr{C}(A)$.
   Conversely, let $M = \begin{pmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{pmatrix} \in \mathscr{C}(A)$. We have $AM = \begin{pmatrix} a & b & c \\ 2x & 2y & 2z \\ 3\alpha & 3\beta & 3\gamma \end{pmatrix}$ and $MA = \begin{pmatrix} a & 2b & 3c \\ x & 2y & 3z \\ \alpha & 2\beta & 3\gamma \end{pmatrix}$.
   Since $AM = MA$ that is $\begin{pmatrix} a & b & c \\ 2x & 2y & 2z \\ 3\alpha & 3\beta & 3\gamma \end{pmatrix} = \begin{pmatrix} a & 2b & 3c \\ x & 2y & 3z \\ \alpha & 2\beta & 3\gamma \end{pmatrix}$ we get $\begin{cases} b = 2b & , \ 2x = x \\ c = 3c & , \ 3\alpha = \alpha \\ 2z = 3z & , \ 3\beta = 2\beta \end{cases}$, hence $b = x = c = \alpha = z = \beta = 0$. It follows that $M = \begin{pmatrix} a & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ is a diagonal matrix. Thus $\mathscr{C}(A) = \{D \in \mathscr{M}_3(\mathbb{R}) / \ D \text{ diagonale } \}$
   Soit $P \in \mathscr{M}_3(\mathbb{R})$ une matrice inversible et $B = PAP^{-1}$, enfin $\mathscr{C}(B) = \{M \in \mathscr{M}_3(\mathbb{R}) / \ BM = MB\}$.

4. $M \in \mathscr{C}(B) \Leftrightarrow MB = BM \Leftrightarrow MPAP^{-1} = PAP^{-1}M \Leftrightarrow P^{-1}MPA = AP^{-1}MP \Leftrightarrow P^{-1}MP \in \mathscr{C}(A)$.

5. $M \in \mathscr{C}(B)$ if and only if $P^{-1}MP \in \mathscr{C}(A) = \{D \in \mathscr{M}_3(\mathbb{R}) / \ D \text{ diagonale } \}$. Hence $M \in \mathscr{C}(B)$ if and only if there exists a diagonal matrix D such that $P^{-1}MP = D$ that is $M = PDP^{-1}$. Thus $\mathscr{C}(B) = \{PDP^{-1} / \ D \text{ diagonale}\}$

Exercice 2

Soit l'espace vectoriel $E=\mathbb{R}^3$, et $f: E \to E$ l'application définie par

$$f(x,y,z)=(x+2y-2z,y+3z,-y+z)$$

1. To do.
2. $f(e_1)=(1,0,0)=e_1$, $f(e_2)=(2,1,-1)=2e_1+e_2-e_3$ and $f(e_3)=(-2,3,1)=-2e_1+3e_2+e_3$. Hence

$$\mathcal{M}_{\mathcal{B}}(f)=\begin{pmatrix}1&2&-2\\0&1&3\\0&-1&1\end{pmatrix}$$

3. $(x,y,z)\in\ker f \Leftrightarrow (x+2y-2z,y+3z,-y+z) \Leftrightarrow x=y=z=0$. So $\ker f=\{0\}$. It follows that $\mathrm{rg}\, f=\dim E-\dim\ker f=3$.
4. $\mathcal{M}_{\mathcal{B}}(f^2)=\left(\mathcal{M}_{\mathcal{B}}(f)\right)^2=\begin{pmatrix}1&2&-2\\0&1&3\\0&-1&1\end{pmatrix}\begin{pmatrix}1&2&-2\\0&1&3\\0&-1&1\end{pmatrix}=\begin{pmatrix}1&6&2\\0&-2&6\\0&-2&-2\end{pmatrix}$

PROBLÈME

Diagonalisation d'une matrice et applications

Dans tout le problème A désigne la matrice

$$A=\begin{pmatrix}-1&1&0\\1&-2&1\\0&1&-1\end{pmatrix}$$

Soit f l'endomorphisme de $E=\mathbb{R}^3$ *canoniquement associé* à la matrice A.
Notons $\mathcal{B}=(e_1,e_2,e_3)$ la base canonique de E.

Première partie : l'endomorphisme f

1. $f(e_1)=(-1,1,0)$, $f(e_2)=(1,-2,1)$ and $f(e_3)=(0,1,-1)$.
2. $f(x,y,z)=xf(e_1)+yf(e_2)+zf(e_3)=(-x+y,x-2y+z,y-z)$.
3. $f(1,1,1)=(0,0,0)$ and $f(1,-2,1)=(-3,6,-3)$.
4. $f(-1,a,b)=(1,-a,-b) \Leftrightarrow (1+a,-1-2a+b,b-c)=(1,-a,-b) \Leftrightarrow 1+a=1$, $-1-2a+b=-a$, $a-b=-b \Leftrightarrow a=0$, $b=1$.

Deuxième partie : Réduction de A

Dans cette partie, on considère les vecteurs $\varepsilon_1=(1,-2,1)$, $\varepsilon_2=(1,1,1)$ et $\varepsilon_3=(-1,0,1)$.

5. The family $\mathcal{B}'=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ has 3 elements, so it is enough to show that it is free. If $a,b,c\in\mathbb{R}$ such that $a\varepsilon_1+b\varepsilon_2+c\varepsilon_3=0$, then it is easy to show that $a=b=c=0$.
6. $P=\mathcal{M}_{\mathcal{B}}(\mathcal{B}')=\begin{pmatrix}1&1&-1\\-2&1&0\\1&1&1\end{pmatrix}$

7. $${}^tPP = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Denote $T = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The matrix T is invertible and $T^{-1} = \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$. Since ${}^tPP = T$, we get $(T^{-1}\,{}^tP)P = I_3$, hence P is invertible and

$$P^{-1} = T^{-1}\,{}^tP = \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ 2 & 2 & 2 \\ -3 & 0 & 3 \end{pmatrix}$$

8. The matrix D : $f(\varepsilon_1) = f(1,-2,1) = (-3,6,-3) = -3\varepsilon_1$, $f(\varepsilon_2) = 0$ and $f(\varepsilon_3) = f(-1,0,1) = (1,0,-1) = -\varepsilon_3$. Hence

$$D = \mathcal{M}_{\mathcal{B}'}(f) = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

9. By the basis change formula, $A = PDP^{-1}$ (or equivalently $P^{-1}AP = D$)

Troisième partie :
Une application

On considère l'équation matricielle : $(\star)\ \ X^3 = A$.
On cherche à déterminer toutes les matrices $X \in \mathcal{M}_3(\mathbb{R})$ vérifiant $(\star)$.
Soit X une matrice vérifiant $(\star)$. On pose $Y = P^{-1}XP$.

10. $Y^3 = (P^{-1}XP)^3 = P^{-1}X^3P = P^{-1}AP = D$.

11. $YD = YY^3 = Y^4 = Y^3Y = DY$.

12. Denote $Y = \begin{pmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{pmatrix}$. Since $YD = DY$, we get $\begin{pmatrix} -3a & 0 & -c \\ -3x & 0 & -z \\ -3\alpha & 0 & -\gamma \end{pmatrix} = \begin{pmatrix} -3a & -3b & -3c \\ 0 & 0 & 0 \\ -\alpha & -\beta & -\gamma \end{pmatrix}$, so $b = c = x = z = \alpha = \beta = 0$. It follows that Y is a diagonal matrix.

13. Y is a diagonal matrix, hence $Y = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$. We have $Y^3 = D$, hence $a^3 = -3$, $b^3 = 0$ and $c^3 = -1$, so $a = -\sqrt[3]{3}$, $b = 0$ and $c = -1$ (since $a, b, c \in \mathbb{R}$).

$$Y = \begin{pmatrix} -\sqrt[3]{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

14. We have $Y = P^{-1}XP$, hence $X = PYP^{-1} = \frac{1}{6} \begin{pmatrix} -\sqrt[3]{3}-3 & 2\sqrt[3]{3} & -\sqrt[3]{3}+3 \\ 2\sqrt[3]{3} & -4\sqrt[3]{3} & 2\sqrt[3]{3} \\ -\sqrt[3]{3}+3 & 2\sqrt[3]{3} & -\sqrt[3]{3}-3 \end{pmatrix}$

END