Master Mathématiques et applications

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Algebraic Geometry

Module $M236$-S3

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Avant-propos

This course workbook (version $\frac{1}{2}$-course) is intended for students of "Master Mathematics and applications" at higher Normal School-Fez. It offers an incomplete course (without proofs) Algebraic Geometry: Presheaves and sheaves, affine schemes, schemes and morphisms of schemes, quasi-coherent O_X-modules, cohomology.

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Chapter 1

Presheaves and Sheaves

1.1 Presheaves

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Definition 1.1.

Let X be a topological space. A presheaf of abelian groups $\mathscr{F}$ on X is defined by the following data:

1. *For any open U of X, an abelian group $\mathscr{F}(U)$,*
2. *For any inclusion of open $V \subseteq U$, a morphism of groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that :*
   - *(a)* $\mathscr{F}(\emptyset) = 0$,
   - *(b)* ρ_{UU} *is the identity morphism* $\mathscr{F}(U) \to \mathscr{F}(U)$,
   - *(c) If* $W \subseteq V \subseteq U$ *are opens, then* $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A categorical version: Let X be a topological space and define the category $\mathfrak{Top}(X)$:

- The objects are the open subsets of X
- The morphisms are the inclusion, that is to say, $\mathrm{Hom}(V, U) = \emptyset$ if $V \nsubseteq U$ and $\mathrm{Hom}(V, U) = \{i : V \to U\}$ where i is the inclusion if $V \subseteq U$.

A presheaf on X is a contravariant functor from the category $\mathfrak{Top}(X)$ to the category $\mathfrak{Ab}$ of Abelian groups.

We define in the same way a presheaf of rings, sets, or with values in a category $\mathfrak{C}$, replacing in the definition " abelian group" by "ring", "set", or "object of $\mathfrak{C}$".

If $\mathscr{F}$ is a prsheaf on X, The elements of the set $\mathscr{F}(U)$ are called the *sections* of $\mathscr{F}$ over U. The map ρ_{UV} is called the *restriction* map. We will use the notation $s|_V := \rho_{UV}(s)$ if $s \in \mathscr{F}(U)$. Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, .)$.

The direct limit (or inductive limit) of a system of abelian groups: Let $(I, \leq)$ be a filtered partially ordered set (meaning that for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$). An inductive system of abelian groups over I consists of:

- A family of abelian groups $(G_i)_{i\in I}$,
- A family of group homomorphisms $\phi_{ji} : G_i \to G_j$ for each pair (i, j) with $i \leq j$, satisfying the coherence conditions $\phi_{ii} = \mathrm{Id}_{G_i}$ and $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ for all $i \leq j \leq k$.

Now, let H be the subgroup of the group $\oplus_{i\in I} G_i$ generated by all elements of the form $x_i - \phi_{ji}(x_i)$, where $x_i \in G_i$. The direct limit (or inductive limit) of the inductive system G_i denoted $\varinjlim_{i\in I} G_i$ is the group:

$$\varinjlim_{i\in I} G_i := \frac{\bigoplus_{i\in I} G_i}{H}$$

An element of the group $\varinjlim_{i\in I} G_i$ is represented by an element $x_i \in G_i$, and two elements $x_i \in G_i$ and $y_j \in G_j$ represent the same element in the direct limit if and only if there exists $k \geq i, j$ such that $\phi_{ki}(x_i) = \phi_{kj}(x_j)$.

Let $\mathscr{F}$ be a presheaf of abelian groups on X and $x \in X$. The set of open neighborhoods of x is partially ordered by (reverse) inclusion: We say $U' \leq U$ if and only if $U \subseteq U'$. The family $(\mathscr{F}(U))_U$, where $x \in U$ open, is an inductive system of abelian groups. The transition maps in the system are given by the restriction maps of $\mathscr{F}$.

Definition 1.2.

Let $\mathscr{F}$ be a presheaf of abelian groups on X and $x \in X$. The stalk of $\mathscr{F}$ at x is the abelian group

$$\mathscr{F}_x := \varinjlim_{x\in U} \mathscr{F}(U)$$

where the colimit is over the set of open neighborhoods U of x in X.

An element of $\mathscr{F}_x$ is represented by a couple (U,s) where U is an open neighborhood of x and $s \in \mathscr{F}(U)$. Two couples (U,s) and (V,t) represent the same element in $\mathscr{F}_x$ if and only if there exist an open neighborhood W of x such that $W \subseteq U \cap V$ and $s|_W = t|_W$. If $s \in \mathscr{F}(U)$ and $x \in U$, the image of s in $\mathscr{F}_x$ is denoted s_x (The germ of s at x). The map $\mathscr{F}(U) \to \mathscr{F}_x$ defined by $s \mapsto s_x$ is an homomorphism of groups.

Definition 1.3.

Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of abelian groups on X. A morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is a rule which assigns to each open $U \subseteq X$ a morphism of abelian groups $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, compatible with restriction, i.e whenever $V \subseteq U \subseteq X$ are opens, the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathscr{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

is commutative.

The commutativity of the diagram can be written: For $s \in \mathscr{F}(U)$,

$$\varphi(U)(s)|_V = \varphi(V)(s|V)$$

Example : Let $X = \mathbb{R}$. Let $\mathscr{F}$ be the presheaf of real valued continuous functions on X (under addition). Let $\mathscr{G}$ be the presheaf of non vanishing bounded continuous complex-valued functions on X (under multiplication). Let $\varphi : \mathscr{F} \to \mathscr{G}$, where for any open $U \subseteq X$, $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, $f \mapsto e^{if}$. Then φ is a morphism of presheaves of abelian groups.

The category of presheaves on a topological space:

The objects: the objects are presheaves of abelian groups,

The morphisms: the morphisms are morphisms of presheaves of abelian groups, i.e $\mathrm{Hom}(\mathscr{F},\mathscr{G})$ are morphisms of presheaves of abelian groups from $\mathscr{F}$ to $\mathscr{G}$.

Identity: For any presheaves $\mathscr{F}$, $\mathrm{Id}_{\mathscr{F}} : \mathscr{F} \to \mathscr{G}$ is the morphism of presheaves defined by $\mathrm{Id}_{\mathscr{F}}(U) = \mathrm{Id}_{\mathscr{F}(U)}$.

Composision: Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ tree presheaves of abelian groups on X, and $\varphi : \mathscr{F} \to \mathscr{G}$, $\psi : \mathscr{G} \to \mathscr{H}$ be two morphisms of presheaves, the morphism $\psi \circ \varphi : \mathscr{F} \to \mathscr{H}$ is defined by: for any open U of X, $(\psi \circ \varphi)(U) = \psi(U) \circ \varphi(U)$.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves and $x \in X$, then φ induce a morphism of groups on the stalk $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$, where φ_x is defined by: $\varphi_x(s_x) = (\varphi(U)(s))_x$, where s_x is represented by (U, s) ($s \in \mathscr{F}(U)$ and U is an open neighborhood of x). The map φ_x is well defined, indeed, is $s_x = t_x$ where $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(t)$, then there exists an open neighborhood W of x such that $s|_W = t|_W$, hence $\varphi(W)(s|W) = \varphi(W)(t|W)$ that is $\varphi(U)(s)|_W = \varphi(V)(t)|_W$, hence $(\varphi(u)(s))_x = (\varphi(V)(t))_x$.

Let $\mathscr{F}$ be a presheaf of abelian groups. The identity mrophism $\mathrm{Id}_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}$ is definied for every open U, by $(\mathrm{Id}_{\mathscr{F}})(U) = \mathrm{Id}_{\mathscr{F}(U)} : \mathscr{F}(U) \to \mathscr{F}(U)$.

Definition 1.4.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of abelian groups.
We say that φ is injective if for any open U of X, $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.
We say that φ is surjective if, for any open U of X, $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective.
We say that φ is an isomorphism if it admits an inverse, that is to say, if there exists a morphism of presheaves of abelian groups $\psi : \mathscr{G} \to \mathscr{F}$ such that $\varphi \circ \psi = \mathrm{Id}_{\mathscr{G}}$ and $\psi \circ \varphi = \mathrm{Id}_{\mathscr{F}}$.

Proposition 1.5.

Let φ be a morphism of presheaves of abelian groups. Then φ is an isomorphism if it is injective and surjective.

Proof : ....

1.2 Sheaves

Definition 2.1.

A sheaf $\mathscr{F}$ of abelian groups on X is a presheaf of abelian groups which satisfies the following additional property:

1. ***(Uniqueness):*** *Given any open covering $U = \cup_{i \in I} U_i$ and $s, t \in$*

$\mathscr{F}(U)$ such that, for any i , $s|_{U_i} = t|_{U_i}$ then $s = t$,

2. ***(Gluing):*** *Given any open covering $U = \cup_{i\in I} U_i$ and any collection of sections $s_i \in \mathscr{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ there exist a section $s \in \mathscr{F}(U)$ such that $\forall i$, $s|_{U_i} = s_i$.*

Example : Let X be a topological space, for U open of X, denote $\mathscr{C}(U) = \mathscr{C}^0(U,\mathbb{R})$ the set of continuous maps from U to $\mathbb{R}$, and ρ_{UV} the usual restriction. Then $\mathscr{C}$ is a sheaf of abelian groups on X.

Example : Let X be a topological space and $x_0 \in X$ a fixed point. Let G be an abelian group. For U open of X set $\mathscr{F}(U) = \begin{cases} G & \text{if} \quad x_0 \in U \\ 0 & \text{if} \quad x_0 \notin U \end{cases}$, for $V \subseteq U$ opens, $\rho_{UV} = \mathrm{Id}_G$ if $x_0 \in V$ and $\rho_{UV} = 0$ if $x_0 \notin V$. Then $\mathscr{F}$ is a sheaf of abelian groups.

Example : Let X be a topological space and $(G_x)_{x\in X}$ a family of groups indexed by the points x of X. For every open U of X we set $\prod(U) = \prod\limits_{x\in U} G_x$. For $V \subset U$ we define the restriction as follows; for $s = (g_x)_{x\in U}$ its restriction to V is $s|_V = (g_x)_{x\in V}$. Then $\prod$ is a sheaf of abelian groups on X.

Let $\mathscr{U} = \{U_i\}_i$ be a family of open subsets of X, and $U = \cup_i U_i$. Denote $U_{ij} = U_i \cap U_j$. For any presheaf $\mathscr{F}$ of abelian groups on X, we have a complex of abelian groups :

$$C^\bullet(\mathscr{U},\mathscr{F}): \quad 0 \longrightarrow \mathscr{F}(U) \xrightarrow{d_0} \prod_i \mathscr{F}(U_i) \overset{d_1}{\rightrightarrows} \prod_{i,j} \mathscr{F}(U_{i,j})$$

Where $d_0 : s \mapsto (s|_{U_i})_i$ and $d_1 : (s_i)_i \mapsto (s_i|_{U_{i,j}} - s_j|_{U_{i,j}})_{i,j}$

Proposition 2.2.

A presheaf $\mathscr{F}$ is a sheaf if and only if or any family of open subsets $\mathscr{U}$ of X the complex $C^\bullet(\mathscr{U},\mathscr{F})$ is exact.

Proof : Indeed; injectivity of d_0 is equivalent to the condition of *uniqueness*, and the equality $\mathrm{im} d_0 = \ker d_1$ is the *gluing* condition.

Proposition 2.3.

Let $\mathscr{F}$ be a sheaf of abelian groups on X and U an open subset of X. If $s, t \in \mathscr{F}(U)$ such that, for any $x \in U$, $s_x = t_x$ then $t = s$.

Proof : ....

1.3 Morphism of sheaves

A morphism of sheaves (of abelian groups) is a morphism of presheaves (of abelian groups).

We have a category $\mathfrak{Ab}(X)$, where the objects are sheaf of abelian groups, and morphism are morphisms of sheaf of abelian groups.

In any category we have the notion of epimorphism, monomorphism, isomorphism.

Definition 3.1.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups.

1. *We say that φ is injective if, for every open set $U \subseteq X$, the map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

2. *We say that φ is surjective if, for every open set $U \subseteq X$ and every $s \in \mathscr{G}(U)$, there exists an open covering $U = \cap_i U_i$ and sections $t_i \in F(U_i)$ such that $s|_{U_i} = \varphi(U_i)(t_i)$ for all i.*

3. *We say that φ is an isomorphism if there exists a morphism of sheaves of abelian groups $\psi : \mathscr{G} \to \mathscr{F}$ such that $\varphi \circ \psi = \mathrm{Id}_{\mathscr{G}}$ and $\psi \circ = \mathrm{Id}_{\mathscr{F}}$.*

Proposition 3.2.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups. Then

1. *φ is injective if and only if for any $x \in X$, $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective.*

2. *φ is surjective if and only if for any $x \in X$, $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ surjective.*

3. *φ is an isomorphism if and only if for any $x \in X$, $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism.*

Proof : . . . .

Another characterization of injectivity and isomorphism is in the following

Theorem 3.3.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism de sheaves of abelian groups. Then φ is an isomorphism if and only if for any open subset U of X, the morphism $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism.

Proof : . . . .

Note that, if $\varphi : \mathscr{F} \to \mathscr{G}$ is surjecive, then for U open, it is not true that, the morphism $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective. As in the following example:

Example: Let $X = \mathbb{C}^*$, with its usual topology. $\mathscr{F}$ the sheaf of holomorphic functions: $U \mapsto \mathscr{F}(U) :=$ the (additive) group of holomorphic functions on U, and $\mathscr{G}$ that of holomorphic invertible functions: $U \mapsto \mathscr{G}(U) :=$ the (multiplicative) group of holomorphic invertible function on U. Let $\varphi : \mathscr{F} \to \mathscr{G}$ the morphism of sheaves defined by: for any open U and $f \in \mathscr{F}(U)$; $\varphi(U)(f) = e^f$. Note that $\varphi(X) : \mathscr{F}(X) \to \mathscr{G}(G)$ is not surjective (the map $z \mapsto z$ is not an exponential); but φ is surjective; indeed:
Let $x \in X$, and $h_x \in \mathscr{G}_x$ where $h \in \mathscr{G}(V)$, there is an open neighborhood U of x simply connected (a disk for example) such that $U \subseteq V$. Since $h|_U$ is invertible on U, then $\dfrac{h'}{h}$ is holomorphic on U, so h has a primitive f on U (by a reason of simply connected), as $(he^{-f})' = h'e^{-f} - hf'e^{-f} = 0$ on U, so $he^{-f} = \lambda$ non-zero constant on U, hence $h = \lambda e^f = e^{f+\alpha} = e^g$ on U where $\lambda = e^\alpha$ and $g = f + \alpha$. This shows that $\varphi_x(g_x) = h_x$.

Sheafiffication:

Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. In this section we explain how to get the sheafification of the presheaf $\mathscr{F}$, that is a process that turns a presheaf into a sheaf . We will use stalks to describe the sheafification in this case.

Definition 3.4.

Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. A sheaf of abelian groups associated to $\mathscr{F}$ is a sheaf of abelian groups $\mathscr{F}^\dagger$ together with a morphism of presheaves $\theta : \mathscr{F} \to \mathscr{F}^\dagger$ satisfying the universal property:
For any morphism of presheaves of abelian groups $\varphi : \mathscr{F} \to \mathscr{G}$ where $\mathscr{G}$ is a sheaf of abelian groups, there exist a unique morphism of sheaves of abelian groups $\tilde{\varphi} : \mathscr{F}^\dagger \to \mathscr{G}$ such the the following diagram commute,

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^\dagger \arrow[ur, "\tilde{\varphi}"] & \end{tikzcd}$$

The uniqueness (when there exits) of $\mathscr{F}^\dagger$ is an immediate consequence of the universal property. Moreover:

Theorem 3.5.

Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. Then the sheaf $\mathscr{F}^\dagger$ of abelian groups associated to $\mathscr{F}$ exists ant it is unique up to isomorphism. Moreover, for ant $x \in X$, $\theta_x : \mathscr{F}_x \to \mathscr{F}^\dagger_x$ is an isomorphism.

Proof : ....

Remark : If $\mathscr{F}$ is a sheaf, then by the universal property $\mathscr{F}^\dagger \simeq \mathscr{F}$.

Example : Constant sheaf : Let G be an abelian group, then $U \mapsto G$ if $U \neq \emptyset$ and $\emptyset \mapsto 0$ is a presheaf and the associated sheaf is called *constant sheaf* with values in G, and it is noted $\underline{G}_X$ or $\underline{G}$, it is the sheaf of continuous functions with values in G endowed with the discrete topology.

1.4 Subsheaf and quotient sheaf

In the following in this sections, all sheaves are sheaves of abelian groups and morphisms are morphisms of sheaves of abelian groups.

Definition 4.1.

Let $\mathscr{F}'$ and $\mathscr{F}$ be two sheaves on X. We say that $\mathscr{F}'$ is subsheaf of $\mathscr{F}$, if for any open U of X, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$ and the restriction maps of $\mathscr{F}'$ are induced by the restriction maps of $\mathscr{F}'$.

Remark: $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$ if and only if the injection $i : \mathscr{F}' \to \mathscr{F}$ is a morphism of sheaves.

Proposition 4.2.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. The presheaf $U \mapsto \ker(\varphi(U))$ is a sheaf, it is called the kernel of φ, and denoted $\ker\varphi$.

Proof : ....

Remark : If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\ker\varphi$ is a subsheaf of $\mathscr{F}$, and it is the kernel in the sense of category, that is to say : $\varphi \circ i = 0$ and for any morphism $\psi : \mathscr{H} \to \mathscr{F}$, such that $\varphi \circ \psi = 0$, there exist morphism $\psi' : \mathscr{H} \to \ker\varphi$ such that $\psi = i \circ \psi'$.

The presheaves image $U \mapsto \mathrm{Im}(\varphi(U))$ and cokernel $U \mapsto \mathrm{coker}(\mathrm{Im}(U))$ are not in general sheaves, which give sense to the following

Definition 4.3.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. The sheaf associated to the presheaf image is called the sheaf image and it is denoted $\mathrm{im}\varphi$. The sheaf cokernel denoted $\mathrm{coker}\varphi$, is the sheaf associated to the presheaf cokernel.

Remark : If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, and U an open of X, it is not true that $(\mathrm{im}\varphi)(U) = \mathrm{im}(\varphi(U))$, the first term is the group of sections of sheaf $\mathrm{im}\varphi$ over the open U, and the second is the image of the morphism of groups $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$. Moreover:

Theorem 4.4.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. We have the following statements:

1. *Let U be an open of X, and $s \in \mathscr{G}(U)$. Then $s \in (\mathrm{im}\varphi)(U)$ if and only if there exists an open cover $(U_i)_i$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that , for any $i \in I$, $s|_{U_i} = \varphi(U_i)(t_i)$.*

2. *φ is surjective if and only if $\mathrm{im}\varphi = \mathscr{G}$.*

Proof : ....

Quotient sheaves: Let $\mathscr{F}'$ a subsheaf of a sheaf $\mathscr{F}$, then $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$ is a presheaf, but it is not in general a sheaf;

Definition 4.5.

The sheaf associated to the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$ is called the quotient sheaf of $\mathscr{F}$ by $\mathscr{F}'$ and is denoted $\mathscr{F}/\mathscr{F}'$.

Remark Since $\mathscr{F}/\mathscr{F}'$ is the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$ and sheafification conserve the stalk, it follows that, for any $x \in X$, $(\mathscr{F}/\mathscr{F}')_x \simeq \mathscr{F}_x/\mathscr{F}'_x$.

1.5 Direct image and inverse image

In this subsection, we recall some transformations of sheaves via continuous maps of topological spaces.

Definition 5.1.

Let X and Y be two topological space, and $f : X \to Y$ a continuous map.
If $\mathscr{F}$ is a sheaf on the topological space X, then $V \mapsto \mathscr{F}(f^{-1}(V))$ is a sheaf on Y, it is called the direct image of the sheaf $\mathscr{F}$ and is denoted $f_*\mathscr{F}$.
If $\mathscr{G}$ is a sheaf on the topological space Y, the sheaf associated to the presheaf $U \mapsto \varinjlim_{f(U)\subseteq V} \mathscr{G}(V)$ is called the inverse image of the sheaf $\mathscr{G}$

and is denoted $f^{-1}\mathscr{G}$.

Remarks

1. If $V \subseteq U$ are open subsets of Y, the restriction map is defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) \to \mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V)$ (note that $f^{-1}(V) \subseteq f^{-1}(U)$ are open subsets of X).

2. For any $x \in X$, we have $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$.

3. Note that f_* is a covariant functor from the category $\mathfrak{Ab}(X)$ to the category $\mathfrak{Ab}(Y)$. Also f^{-1} is a functor from the category $\mathfrak{Ab}(Y)$ to the category $\mathfrak{Ab}(X)$.

Definition 5.2.

Let Z be a subset of X, together with the induced topology and $i : Z \hookrightarrow X$ the inclusion. If $\mathscr{F}$ is a sheaf on X, the sheaf $i^{-1}\mathscr{F}$ is called the restriction of $\mathscr{F}$ to Z and is denoted $\mathscr{F}|_Z$.

Remark : If U is an open subset of X, then for any open $V \subseteq U$, we have $\mathscr{F}|_U(V) = \mathscr{F}(V)$.

Proposition 5.3.

Let Z be a subset of X and $\mathscr{F}$ a sheaf on X. Then for any $z \in Z$, we have $(\mathscr{F}|_Z)_z = \mathscr{F}_z$

Proof : For $z \in Z$, we have $i(z) = z$, hence $(\mathscr{F}|_Z)_z = (i^{-1}\mathscr{F})_z = \mathscr{F}_{i(z)} = \mathscr{F}_z$.

1.6 Exact sequence of sheaves

Definition 6.1.

Let X be a topological space. A sequence of sheaves of abelian groups on X

$$\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \xrightarrow{\varphi^{i+1}} \cdots$$

is said exact, if for any i; $\mathrm{im}\varphi^{i-1} = \ker\varphi^i$. In particular; a sequence $0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is exact if φ is injective, and the sequence $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \longrightarrow 0$ is exact if φ is surjective.

Example : Let $X = \mathbb{R}$ with its usual topology. We define three sheaves of abelian groups under standard addition:

1. $\mathscr{C}^\infty$: The sheaf of smooth (infinitely differentiable) functions.
2. $\underline{\mathbb{R}}$: The constant sheaf of real numbers (locally constant functions).

Then the sequence

$$0 \longrightarrow \underline{\mathbb{R}} \xrightarrow{\text{inclusion}} \mathscr{C}^\infty \xrightarrow{\frac{d}{dx}} \mathscr{C}^\infty \longrightarrow 0$$

is exact.

Lemma 6.2.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then for any $x \in X$, we have $(\ker\varphi)_x = \ker(\varphi_x)$ and $(\mathrm{im}\varphi)_x = \mathrm{Im}(\varphi_x)$.

Proof : ....

The following theorem is a local characterization of the exactness:

Theorem 6.3.

A sequence of sheaves of abelian groups on X

$$\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \xrightarrow{\varphi^{i+1}} \cdots$$

is exact if and only if for any $x \in X$, the sequence of abelian groups

$$\cdots \longrightarrow \mathscr{F}_x^{i-1} \xrightarrow{\varphi_x^{i-1}} \mathscr{F}_x^{i} \xrightarrow{\varphi_x^{i}} \mathscr{F}_x^{i+1} \xrightarrow{\varphi_x^{i+1}} \cdots$$

is exact.

Proof : ....

An immediate consequence, is the following

Proposition 6.4.

Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Then the sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$ is exact. Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves on X, then $\mathscr{F}'$ is identified to a subsheaf of $\mathscr{F}$ and $\mathscr{F}'' \simeq \mathscr{F}/\mathscr{F}'$.

Proof : ....

Corollary 6.5.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then

1. $\mathrm{im}(\varphi) \simeq \mathscr{F}/\ker\varphi$.
2. $\mathrm{coker}\varphi \simeq \mathscr{G}/\mathrm{im}(\varphi)$.

Proof : ....

One can easily show that, the $\ker$ and Coker are the categorical kernel and cokernel.

Theorem 6.6. (Birth of the Cohomology)

Let U be an open subset of X. The global sections functor $\Gamma(U, .)$ over U, from the category of $\mathfrak{Ab}(X)$ to the category $\mathfrak{Ab}$ is left exact, that is to say; if $0 \to \mathscr{F} \to \mathscr{F}' \to \mathscr{F}''$ is an exact sequence of sheaves of abelian groups on X, then the sequence $0 \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}'')$ is exact.

Proof : ....

Example : Let $\mathscr{H}$ the sheaf of (additive) abelian groups of holomorphics functions and $\mathscr{H}^\times$ the sheaves of (multiplicative) abelian groups of invertible holomorphic functions.

Any holomorphic invertible function is locally a logarithm of holomorphic function. By the way, the exponential of a holomorphic function f on an open U of $\mathbb{C}$ is equal to 1 if and only if, f is constant with values belonging to $2i\pi\mathbb{Z}$ on each connected component of U. So we have an exact sequence of sheaves of Abelian groups on the topological space $\mathbb{C}$:

$$0 \to \underline{2i\pi\mathbb{Z}} \to \mathscr{H} \to \mathscr{H}^{\times} \to 1$$

But the sequence $0 \to \underline{2i\pi\mathbb{Z}}(\mathbb{C}^*) = 2i\pi\mathbb{Z} \to \mathscr{H}(\mathbb{C}^*) \to \mathscr{H}^{\times}(\mathbb{C}^*) \to 1$, is not exact: the equality $\underline{2i\pi\mathbb{Z}}(\mathbb{C}^*) = 2i\pi\mathbb{Z}$ is a consequence of the fact that $\mathbb{C}^*$ is connected, and the non-surjectivity of the last morphism is treated in example of a previous section.

This lack of exactness, the precise measure of which is the object of what is called *cohomology*, is in a sense the main interest of sheaf theory: it indeed reflects the difficulties of gluing.

Extension of sheaf by zero : Let X be a topological space, Z a closed subset of X, set $U = X \setminus Z$, and let $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ be inclusions.

• If $\mathscr{H}$ is a sheaf on U, denote $j_!\mathscr{H}$ the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$ and $V \mapsto 0$ else.

Proposition 6.7.

Let X be a topological space, U an open of X and set $Z = X \setminus U$.

1. *If $\mathscr{H}$ is a sheaf on Z. Then for any $x \in X$, we have $(i_*\mathscr{H})_x = \mathscr{H}_x$ if $x \in Z$ and $(i_*\mathscr{H})_x = 0$ else.*

2. *If $\mathscr{K}$ is a sheaf on U. Then for any $x \in X$, we have $(j_!\mathscr{K})_x = \mathscr{K}_x$ if $x \in U$ and $(j_!\mathscr{K})_x = \mathscr{K}_x = 0$ else.*

The sheaf $j_!\mathscr{K}$ is called the extension of the sheaf $\mathscr{K}$ by zero.

Proof : ....

Theorem 6.8.

Let $\mathscr{F}$ be a sheaf on X, Z a closed subset of X and $U = X \setminus Z$, then

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

is an exact sequence of sheaves of abelian groups on X.

Proof : ....

Corollary 6.9.

Let X be a topological space, U an open of X and set $Z = X \setminus U$. If $\mathscr{F}$ is a sheaf on X such that $\mathscr{F}|_U = 0$, then $\mathscr{F} \simeq i_*(\mathscr{F}|_Z)$.

Proof : ....

1.7 Ringed topological space

1.7.1 Ringed topological space

Definition 7.1.

A ringed topological space is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of unitary commutative rings on X, called the structural sheaf.

Examples :

1. Let X be a topological space, and $\mathscr{O}_X$ the sheaf of continuous functions with real values on X, then $(X, \mathscr{O}_X)$ is a ringed space.
2. Let X be a complex analytic manifold (or differential) and $\mathscr{O}_X$ the sheaf of holomorphic functions ($\mathscr{C}^\infty$) over X, then $(X, \mathscr{O}_X)$ is a ringed topological space.
3. Let X be a topological space, then $(X, \underline{\mathbb{Z}}_X)$ is a ringed topological space.
4. Let $(X, \mathscr{O}_X)$ be a ringed topological space, and U be an open of X, then $(U, \mathscr{O}_X|_U)$ is a ringed topological space, the structural sheaf $\mathscr{O}_X|_U$ denoted $\mathscr{O}_U$.

Definition 7.2.

A morphism of ringed topological spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves of rings on Y.

Example : Let $(X,\mathcal{O}_X)$ be a ringed topological space and U be an open of X, set $j:U\to X$ the injection. Then $(j,j^*):(U,\mathcal{O}_U)\to(X,\mathcal{O}_X)$ is a morphism of ringed topological spaces, where for any open V of X, $j^*(V):\mathcal{O}_X(V)\to\mathcal{O}_U(U\cap V)$ is the restriction morphism.

Definition 7.3.

Let $(f,f^\sharp):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ and $(g,g^\sharp):(Y,\mathcal{O}_Y)\to(Z,\mathcal{O}_Z)$ two morphisms of ringed topological spaces. The composition is defined by:

$$(g,g^\sharp)\circ(f,f^\sharp):=(g\circ f,f^\sharp\circ g^\sharp)$$

An isomorphism of ringed topological spaces is a morphism that has an inverse.

1.7.2 $\mathcal{O}_X$-modules

Definition 7.4.

Let $(X,\mathcal{O}_X)$ a ringed topological space. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{M}$ on X such that, for any open U, $\mathcal{M}(U)$ is a $\mathcal{O}_X(U)$-module and for any pair of opens $V\subset U$, the following diagram

$$\begin{array}{ccc}\mathcal{O}_X(U)\times\mathcal{M}(U) & \longrightarrow & \mathcal{M}(U)\\ \downarrow & & \downarrow\\ \mathcal{O}_X(V)\times\mathcal{M}(V) & \longrightarrow & \mathcal{M}(V)\end{array}$$

commute i.e for any $(a,s)\in\mathcal{O}_X(U)\times\mathcal{M}(U)$, we have $(as)|_V=a|_V s|_V$.

Remark : If $\mathcal{M}$ is an $\mathcal{O}_X$-module and $x\in X$, then $\mathcal{M}_x$ has a natural structure of $\mathcal{O}_{X,x}$-module: $(a_x)(m_x):=(am)_x$.

Example : Let X be a topological space. Then $\underline{\mathbb{Q}}$ is a $\underline{\mathbb{Z}}$-module.

Definition 7.5.

Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{O}_X$-modules. A morphism $\varphi:\mathcal{M}\to\mathcal{N}$ is morphism of $\mathcal{O}_X$-modules, if it is a morphism of sheaves of abelian groups *such that, for any open U of X, the following diagram*

$$\begin{tikzcd} \mathscr{O}(U)\times\mathscr{M}(U) \arrow[r] \arrow[d, "\mathrm{Id}\times\varphi(U)"] & \mathscr{M}(U) \arrow[d, "\varphi(U)"] \\ \mathscr{O}(U)\times\mathscr{N}(U) \arrow[r] & \mathscr{N}(U) \end{tikzcd}$$

commute, i.e for any $(a,s)\in\mathscr{O}(U)\times\mathscr{F}(U)$, we have $\varphi(U)(as)=a\varphi(U)(s)$, in particular $\varphi(U)(as)|_V = a|_V\varphi(V)(s|_V)$.

Example : Any sheaf of abelian groups $\mathscr{F}$ on a topological space X has a natural structure of $\underline{\mathbb{Z}}_X$-module, and any morphism of sheaves of abelian groups is a morphism of $\underline{\mathbb{Z}}_X$-modules.

Remark : If $\varphi:\mathscr{F}\to\mathscr{G}$ is a morphism of $\mathscr{O}_X$-modules, and $x\in X$, then φ_x has a natural structure of $\mathscr{O}_{X,x}$-morphism: $\varphi_x(a_xm_x)=a_x\varphi_x(m_x)$.

<u>Direct sum :</u>

Let $\mathscr{M}$ and $\mathscr{N}$ be two sheaves of $\mathscr{O}_X$-modules, then $U\mapsto\mathscr{M}(U)\oplus\mathscr{N}(U)$ define a sheaf of $\mathscr{O}_X$-modules on X called the direct sum of $\mathscr{M}$ and $\mathscr{N}$ and denoted by $\mathscr{M}\oplus\mathscr{N}$.

Theorem 7.6.

Let $(X,\mathscr{O}_X)$ be a ringed space, $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{O}_X$-modules on X, then for any $x\in X$, we have $(\mathscr{M}\oplus\mathscr{N})_x=\mathscr{M}_x\oplus\mathscr{N}_x$.

Proof : ....

<u>Tensor product:</u>

Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{O}_X$-modules, then $U\mapsto\mathscr{M}(U)\otimes_{\mathscr{O}(U)}\mathscr{N}(U)$ defines a presheaf of $\mathscr{O}_X$-modules on X, the associated sheaf is called the tensor product of $\mathscr{M}$ and $\mathscr{N}$ and is denoted $\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N}$.

Theorem 7.7.

Let $(X,\mathscr{O}_X)$ be a ringed space, $\mathscr{M}$ and $\mathscr{N}$ two $\mathscr{O}_X$-modules on X, for any $x\in X$, we have $(\mathscr{M}\otimes_{\mathscr{O}}\mathscr{N})_x=\mathscr{M}_x\otimes_{\mathscr{O}_x}\mathscr{N}_x$.

If $\mathscr{M}$ and $\mathscr{N}$ are two sheaves of abelian groups on X (i.e $\underline{\mathbb{Z}}_X$-modules), the tensor product of $\mathscr{M}$ and $\mathscr{N}$ is their tensor product as $\underline{\mathbb{Z}}_X$-modules $\mathscr{M}\otimes\mathscr{N} := \mathscr{M}\otimes_{\underline{\mathbb{Z}}_X}\mathscr{N}$, in particular for any $x\in X$, $(\mathscr{F}\otimes\mathscr{G})_x = \mathscr{F}_x\otimes_{\mathbb{Z}}\mathscr{G}_x$.

1.7.3 Locally ringed space

Definition 7.8.

A local ringed space or geometric space , is a ringed topological $(X,\mathscr{O}_X)$ such that for all $x\in X$, $\mathscr{O}_{X,x}$ is a local ring . If we denote by $m_{X,x}$ the maximal ideal of $\mathscr{O}_{X,x}$ the field $k(x) := \mathscr{O}_{X,x}/m_{X,x}$ is called the residual field at x.

Examples :

1. Let X be a complex analytic manifold, and $\mathscr{O}_X$ be the sheaf of holomorphic functions on X, so $(X,\mathscr{O}_X)$ is a geometric space.
2. Even for a differential manifold, with the sheaf of $\mathscr{C}^\infty$ functions.
3. Let $(X,\mathscr{O}_X)$ a geometric space, and U an open X, then $(U,\mathscr{O}_U)$ is a geometric space: it is a consequence of the fact that for every $x\in U$, $\mathscr{O}_{U,x} = \mathscr{O}_{X,x}$.

Let $(X,\mathscr{O}_X)$ be a local ringed space, $x\in X$, we have a canonical surjection $\mathscr{O}_{X,x}\to k(x)$, let's note this surjection $f\mapsto f(x)$ called *evaluation* in x. We thus have the equivalence $f(x)\neq 0$ if and only if f is invertible in $\mathscr{O}_{X,x}$. Let U be an open of X and $x\in U$, the composed morphism $\mathscr{O}_X(U)\to\mathscr{O}_{X,x}\to k(x)$ will be noted as $f\mapsto f(x)$, and note that if f is an invertible element of $\mathscr{O}_X(U)$, then $f(x)$ is a nonzero element $k(x)$.

Theorem 7.9.

Let $(X,\mathscr{O}_X)$ be a local ringed space, U open of X, and $f\in\mathscr{O}_X(U)$.
The set $D(f) := \{x\in U \ \ / \ f(x)\neq 0\}$ is an open of U, and f is invertible in $\mathscr{O}_X(U)$ if and only if $D(f) = U$.

Proof : ....

Definition 7.10.

Let A and B be two local rings, and denote m_A (resp. m_B) the maximal ideal of A (resp. of B), and let $\varphi : A \to B$ be a morphism of rings. We say that φ is local if $\varphi(m_A) \subseteq m_B$.

Proposition 7.11.

With the notations of the previous definition. The following statements are equivalents:

1. *φ is local ,*
2. *$m_A \subseteq \varphi^{-1}(m_B)$,*
3. *$m_A = \varphi^{-1}(m_B)$,*
4. *For any $x \in A$, if $\varphi(x)$ is invertible in B, then x is invertible in A.*

Example : Let $\varphi : A \to B$ be a morphism of rings, and Q a prime ideal of B, $P = \varphi^{-1}(Q)$, then the induced morphism $A_P \to B_Q$ is local.

Definition 7.12.

A morphism of local ringed spaces is a morphism of ringed topological spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for any $x \in X$, the morphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local, i.e $f_x^\sharp(m_{Y,f(x)}) \subseteq m_{X,x}$.

Remark : Let $x \in X$, say that $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local, this means that, if g is an element of $\mathcal{O}_{Y,f(x)}$, so g belongs to the maximal ideal of $\mathcal{O}_{Y,f(x)}$ if and only if $f_x^\sharp(g)$ belongs to the maximal ideal of $\mathcal{O}_{X,x}$. This translates to the following equivalence:

$$g(f(x)) = 0 \Leftrightarrow \left[f_x^\sharp(g) \right](x) = 0$$

1.8 Exercises

Exercise 1.1

Let $X = \mathbb{R}$ with the standard topology. For every open set $U \subseteq \mathbb{R}$, let $\mathscr{F}(U)$ be the abelian group of all bounded continuous real-valued functions under pointwise addition. The restriction maps are standard function restrictions.

1. *Show that $\mathscr{F}$ is a presheaf of abelian groups.*
2. *Prove that $\mathscr{F}$ satisfies the uniqueness axiom.*
3. *Prove that $\mathscr{F}$ fails the gluing axiom.*

Exercise 1.2

Let X be a topological space, $x_0 \in X$ a fixed point, and G an abelian group. Let $\mathscr{F}$ be the skyscraper sheaf defined by:

$$\mathscr{F}(U) = \begin{cases} G & \text{if } x_0 \in U \\ 0 & \text{if } x_0 \notin U \end{cases}$$

1. *Compute the stalk $\mathscr{F}_{x_0}$.*
2. *Assume that x_0 is a closed point. Compute $\mathscr{F}_x$ for $x \in X$.*

Exercise 1.3

Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. For any open set $U \subseteq X$ and any section $s \in \mathscr{F}(U)$, the support of s is defined as: $supp(s) = \{x \in U \mid s_x \neq 0 \text{ in } \mathscr{F}_x\}$ where s_x is the germ of s at x.

1. *Prove that the complement $U \setminus supp(s)$ is an open subset of U.*
2. *Let $s, t \in \mathscr{F}(U)$, prove that $supp(s+t) \subseteq supp(s) \cup supp(t)$.*

Exercise 1.4

A sheaf of abelian groups $\mathscr{F}$ on X is called a torsion sheaf if every stalk $\mathscr{F}_x$ is a torsion abelian group (i.e., every element in $\mathscr{F}_x$ has finite order).

1. *Prove that if $\mathscr{F}$ is a torsion sheaf, then for any open set $U \subseteq X$ and any global section $s \in \mathscr{F}(U)$, the section s is locally torsion (for every $x \in U$, there exists a neighborhood V_x and an integer $n_x > 0$ such that $n_x \cdot s|_{V_x} = 0$).*
2. *If U is a qusi-compact open subset and $s \in \mathscr{F}(U)$, prove that there exists an integer $n > 0$ such that $ns = 0$ on U.*

Exercise 1.5

Let $X = [0,1]$ with the standard topology. Define a presheaf $\mathscr{F}$ on X by setting $\mathscr{F}(U) = \mathbb{Z}$ if $U = X$, and $\mathscr{F}(U) = 0$ for any other open set $U \subsetneq X$. The restriction maps are forced to be zero maps except $\mathscr{F}(X) \to \mathscr{F}(X)$ which is the identity.

1. *Show that $\mathscr{F}$ is a presheaf of abelian groups.*
2. *Prove that for every point $x \in X$, the stalk $\mathscr{F}_x = 0$.*
3. *What is the sheafification of $\mathscr{F}$?*

Exercise 1.6

Let $X = \mathbb{R}$ with the standard topology. Consider the constant presheaf $\underline{\mathbb{Z}}_{pre}$ and the constant sheaf $\underline{\mathbb{Z}}$.

1. *If $U \subseteq \mathbb{R}$ is a connected open, show that $\underline{\mathbb{Z}}(U) = \mathbb{Z}$.*
2. *What is $\underline{\mathbb{Z}}(\mathbb{R}^*)$?*
3. *Find an open set $U \subseteq \mathbb{R}$ where $\underline{\mathbb{Z}}_{pre}(U) \neq \underline{\mathbb{Z}}(U)$.*

Exercise 1.7

Let $f : X \to Y$ be a continuous map. Let

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be an exact sequence of abelian sheaves on X. Prove that the sequence of direct image sheaves on Y:

$$0 \longrightarrow f_*\mathscr{F}' \longrightarrow f_*\mathscr{F} \longrightarrow f_*\mathscr{F}''$$

is exact.

Exercise 1.8

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X. We define the support of φ as the set of points where the map on stalks is non-zero: $supp(\varphi) = \{x \in X \mid \varphi_x \neq 0\}$. Prove that if $supp(\varphi) = \emptyset$, then $\varphi = 0$.

Exercise 1.9

Let $(X, \mathscr{O}_X)$ be a locally ringed space, and let $Y \subseteq X$ be a closed subspace. For U open of X, set $\mathscr{J}_Y(U) = \left\{f \in \mathscr{O}_X(U) \,\middle|\, \forall x \in Y \cap U,\ f_x \in \mathfrak{m}_x\right\}$.

1. *Prove that for every open set $U \subseteq X$, $\mathscr{I}_Y(U)$ is an ideal of the ring $\mathscr{O}_X(U)$.*

2. *Show that $\mathscr{I}_Y$ is a sub-$\mathscr{O}_X$-module of $\mathscr{O}_X$, that is an ideal of $\mathscr{O}_X$.*

Exercise 1.10

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces (meaning $f : X \to Y$ is continuous and we have a ring morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$). Let $\mathscr{M}$ be an $\mathscr{O}_X$-module.

1. *Show that the direct image sheaf $f_*\mathscr{M}$ is naturally a sheaf of abelian groups on Y.*

2. *Use $f^\sharp$ to define a ring action that turns $f_*\mathscr{M}$ into a valid $\mathscr{O}_Y$-*

module on Y.

Exercise 1.11

Let $i : Z \hookrightarrow X$ be the inclusion of a closed subset Z into X. Let $\mathscr{F}$ be a sheaf of abelian groups on Z. Define a presheaf $i_!\mathscr{F}$ on X by $i_!\mathscr{F}(U) = \mathscr{F}(U \cap Z)$.

1. *Prove that $i_!\mathscr{F}$ is a sheaf.*
2. *Show that the stalk of this sheaf satisfies $(i_!\mathscr{F})_x = \mathscr{F}_x$ if $x \in Z$, and 0 if $x \notin Z$.*

Exercise 1.12

A sheaf $\mathscr{F}$ is flabby (flasque) if all its restriction maps are surjective.

1. *Let $X = \mathbb{R}$ with the discrete topology, and let $\mathscr{F} = \underline{\mathbb{Z}}$. Prove that $\mathscr{F}$ is flabby.*
2. *If $X = \mathbb{R}$ with the standard topology, prove that the constant sheaf $\underline{\mathbb{Z}}$ is not flabby.*

Exercise 1.13

Let $(X, \mathscr{O}_X)$ be a ringed space where $\mathscr{O}_X$ is a sheaf of integral domains. For an $\mathscr{O}_X$-module $\mathscr{M}$, define the presheaf

$$\mathscr{T}(U) = \{m \in \mathscr{M}(U) \mid \exists f \in \mathscr{O}_X(U) \setminus \{0\},\ f \cdot m = 0\}$$

1. *Prove that $\mathscr{T}$ is a presheaf of sub-$\mathscr{O}_X$-module.*
2. *Prove that $\mathscr{T}$ is a sheaf.*

Exercise 1.14

Let $\mathscr{M}$ be an $\mathscr{O}_X$-module. We define the annihilator sheaf $\mathrm{Ann}(\mathscr{M})$ *as the sheaf of ideals whose sections annihilate $\mathscr{M}$ locally:*

$$Ann(\mathscr{M})(U) = \{a \in \mathscr{O}_X(U) \mid \forall V \subseteq U \text{ open}, \ (a|_V) \cdot m = 0, \ \forall m \in \mathscr{M}(V)\}$$

1. *Prove that* $\mathrm{Ann}(\mathscr{M})$ *is an $\mathscr{O}_X$-module.*
2. *Show that the stalk* $\mathrm{Ann}(\mathscr{M})_x$ *is contained in* $\mathrm{Ann}(\mathscr{M}_x)$.

Exercise 1.15

Let $(X,\mathscr{O}_X)$ be a ringed space, $\mathscr{M}$ a sheaf of $\mathscr{O}_X$-modules, and $s \in \mathscr{M}(X)$ a global section. Let $\phi : \mathscr{O}_X \to \mathscr{M}$, $\phi_U(a) = a.s|_U$.

1. *Prove that ϕ is a morphism of $\mathscr{O}_X$-modules.*
2. *Let $x \in X$. Show that* $(\mathrm{Im}\,\phi)_x = s_x \mathscr{O}_{X,x}$.

Exercise 1.16

Let $(X,\mathscr{O}_X)$ be a ringed topological space, $\mathscr{M}$ be an $\mathscr{O}_X$-module, and $s \in \mathscr{M}_X(X)$. For U open of X, set $\mathscr{F}(U) = s|U\mathscr{O}_X(U) = \{a.s|_U \ / a \in \mathscr{O}_X(U)\}$.

1. *Show that $\mathscr{F}$ is a presheaf of $\mathscr{O}_X$-module.*
2. *Let $\mathscr{F}'$ be the sheaf associated to $\mathscr{F}$ and $x \in X$, show that $\mathscr{F}'_x = \mathscr{O}_{X,x} s_x$.*

Exercise 1.17

Let $(X,\mathscr{O}_X)$ be a ringed space. We assume that $\mathscr{O}_X$ is a sheaf of integral domains, which means that for every open set $U \subseteq X$, the ring of sections $\mathscr{O}_X(U)$ is an integral domain. We define the presheaf of fractions $\mathscr{K}_X^{pre}$ by setting:

$$\mathscr{K}_X^{pre}(U) = Rrac(\mathscr{O}_X(U)) = \left\{ \frac{f}{g} \;\middle|\; f \in \mathscr{O}_X(U),\, g \in \mathscr{O}_X(U) \setminus \{0\} \right\}$$

1. *Justifies that $\mathscr{K}_X^{pre}$ is a presheaf of rings.*

Let $\mathscr{K}_X$ be the sheaf associated to $\mathscr{K}_X^{pre}$.

2. *Let $x \in X$, prove that $(\mathscr{K}_X)_x \cong Frac(\mathscr{O}_{X,x})$.*
3. *Let $\iota : \mathscr{O}_X \to \mathscr{K}_X$ be the morphism of sheaves induced by the moprhism of presheaves $\mathscr{O}_X \to \mathscr{K}_X^{pre}$. Show that ι is injective.*

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