

Corrigé

Commutative algebra

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédaction.

B

A

MMA

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

صدق الله العظيم

وَعَلَى اللَّهِ فَلْيَتَوَكَّلِ الْمُتَوَكِّلُونَ

Exercice 1

Let R be a ring such that for any $x \in R$, there exists $n \geq 2$ such that $x^n = x$. Show that every prime ideal of R is maximal.

☞ :

Let P be a prime ideal of R . Let $\bar{x} \in R/P$ be a nonzero element, then $\bar{x}^n = \bar{x}$, hence $\bar{x}(\bar{x}^{n-1} - \bar{1}) = \bar{0}$. Since R/P is an integral domain and $\bar{x} \neq \bar{0}$, we get $\bar{x}^{n-1} - \bar{1} = \bar{0}$ that is $\bar{x}^{n-1} = \bar{1}$. It follows that \bar{x} is invertible. Hence R/P is a field. Thus P is a maximal ideal of R .

Exercice 2

Let R be a ring, $(P_\alpha)_{\alpha \in \Gamma}$ be a family of prime ideals of R , and $I = \bigcap_{\alpha \in \Gamma} P_\alpha$. Show that $\sqrt{I} = I$.

☞ :

Let $x \in \sqrt{I}$. there exists $n \in \mathbb{N}$ such that $x^n \in I = \bigcap_{\alpha \in \Gamma} P_\alpha$. That is for each $\alpha \in \Gamma$, $x^n \in P_\alpha$. But P_α is a prime ideal, this yields $x \in P_\alpha$. Therefore $x \in \bigcap_{\alpha \in \Gamma} P_\alpha = I$.

Exercice 3

Let N be a submodule of a module M over a ring R . Show that :

1. If N and M/N are finitely generated, then so is M .
2. If M is finitely generated, then so is M/N .

☞ :

1. Since N and M/N are finitely generated, there exist $l_1, \dots, l_r \in N$ and $m_1, \dots, m_s \in M$ such that $N = (l_1, \dots, l_r)$ and $M/N = (\bar{m}_1, \dots, \bar{m}_s)$. We show that $M = (l_1, \dots, l_r, m_1, \dots, m_s)$. It is clear that $(l_1, \dots, l_r, m_1, \dots, m_s) \subseteq M$. Now, let $m \in M$. There exist $a_1, \dots, a_s \in R$ such that $\bar{m} = a_1 \bar{m}_1 + \dots + a_s \bar{m}_s$, hence $m - (a_1 m_1 + \dots + a_s m_s) \in N$. So there exist $b_1, \dots, b_r \in R$ such that $m - (a_1 m_1 + \dots + a_s m_s) = b_1 l_1 + \dots + b_r l_r$, that is $m = a_1 m_1 + \dots + a_s m_s + b_1 l_1 + \dots + b_r l_r$.

2. If M is finitely generated then $M = (m_1, \dots, m_n)$ where $m_i \in M$. it is easy to see that $M/N = (\overline{m_1}, \dots, \overline{m_n})$.

Exercice 4

1. Show that $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}) = 0$.
2. Let R be ring and $f \in R$. Show that $R_f \otimes_R (R/(f)) = 0$

☞ :

1. For $n, m \in \mathbb{Z}$, we have $\overline{n} \otimes \overline{m} = (6-5)\overline{a} \otimes \overline{b} = 5\overline{a} \otimes \overline{b} - 6\overline{a} \otimes \overline{b} = \overline{5a} \otimes \overline{b} - \overline{a} \otimes \overline{6b} = 0$.
2. $\frac{a}{f^n} \otimes \overline{b} = \frac{fa}{f^{n+1}} \otimes \overline{b} = \frac{a}{f^{n+1}} \otimes f\overline{b} = 0$.

Exercice 5

Let R be a ring and M be an R -module. Let $f_1, \dots, f_n \in R$ such that $(f_1, \dots, f_n) = R$. Prove that the morphisme

$$M \rightarrow \prod_{i=1}^n M_{f_i}$$

is injective.

☞ :

Let $m \in M$ such that its image in $\prod_{i=1}^n M_{f_i}$ is zero. That is for each i , $\frac{m}{1} = 0$ in M_{f_i} . Hence for each i , there exists $l_i \in \mathbb{N}$ such that $f_i^{l_i} m = 0$. Now, we show that $(f_1^{l_1}, \dots, f_n^{l_n}) = R$. By contrapositive, assume that $(f_1^{l_1}, \dots, f_n^{l_n}) \neq R$, then there exists a prime ideal P such that $(f_1^{l_1}, \dots, f_n^{l_n}) \subseteq P$, in particular, each $f_i^{l_i} \in P$, hence $f_i \in P$, that is $(f_1, \dots, f_n) \subseteq P$ which is not possible since $(f_1, \dots, f_n) = R$. Now, we have $(f_1^{l_1}, \dots, f_n^{l_n}) = R$, so there exist $a_1, \dots, a_n \in R$ such that $1 = a_1 f_1^{l_1} + \dots + a_n f_n^{l_n}$. So that $m = m1 = a_1 f_1^{l_1} m + \dots + a_n f_n^{l_n} m = 0$.