Examen : Corrigé

Commutative algebra

Il sera tenu compte, dans l'appréciation des copies, de la précision des raisonnements ainsi que la clarté de la rédaction.

MMA

بِسمِ اللّهِ الرَّحمٰنِ الرَّحِيمِ

وَ علَى اللَّهِ فَليَتَوَكَّلِ المُتَوَكِّلون

صدق الله العظيم

Exercice 1

Let R be a ring and I be an ideal of R.

1. Show that $\sqrt{I}$ is an ideal of R.
2. If P is a prime ideal of R, show that $\sqrt{P}=P$.

☞ :

1. Since $0^1=0\in I$, $0\in\sqrt{I}$.
   If $x\in\sqrt{I}$ and $a\in R$, then $x^n\in I$ for some $n\in N$, so $(ax)^n=a^nx^n\in I$, hence $ax\in\sqrt{I}$.
   Let $a,b\in\sqrt{I}$. Then there exist $n\in\mathbb{N}$ such that $a^n,b^n\in I$. We have $(a+b)^{2n}=\sum_{k=0}^{2n}C_{2n}^k a^k b^{2n-k}$.
   For $k\geq n$, we see that $a^k\in I$, hence $C_{2n}^k a^k b^{2n-k}\in I$. For $0\leq k\leq n$, we see that $b^{2n-k}\in I$ since $2n-k\geq n$, hence $C_{2n}^k a^k b^{2n-k}\in I$. Thus $(a+b)^{2n}\in I$. So $a+b\in I$.
2. Clearly $P\subseteq\sqrt{P}$. Let $x\in\sqrt{P}$, then $x^n\in P$ for some $n\in\mathbb{N}$. Since P is a prime ideal, we get $x\in P$. Thus $\sqrt{P}=P$.

Exercice 2

Let $n\in\mathbb{N}^*$. Show that $\mathbb{Q}\otimes_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})=0$.

☞ :

Let $r\in\mathbb{Q}$ and $m\in\mathbb{Z}$. Then $r\otimes\overline{m}=\frac{nr}{n}\otimes\overline{m}=\frac{r}{n}\otimes\overline{nm}=0$.

Exercice 3

Let M be an R-module and S be a multiplicatively closed subset of R.

1. Show that every element of $S^{-1}R\otimes_R M$ has the form $\frac{a}{s}\otimes m$, where $a\in R$, $s\in S$, $m\in M$.

2. Show that $S^{-1}R\otimes_R M\simeq S^{-1}M$.

☞ :

1. Let $z\in S^{-1}R\otimes_R M$. Then $z=\sum_{i=1}^{n}\frac{a_i}{s_i}\otimes m_i$ where $a_i\in R$, $s_i\in S$, $m_i\in M$. For each i, we have $\frac{a_i}{s_i}\otimes m_i=\frac{1}{s_i}\otimes(a_im_i)$. Denote $s=\prod_{k=1}^{n}s_k$ and $t_i=\prod_{k=1,k\neq i}^{n}s_k$. Then $\frac{a_i}{s_i}\otimes m_i=\frac{t_i}{s}\otimes(a_im_i)=\frac{1}{s}\otimes(t_ia_im_i)$. Hence $z=\frac{1}{s}\otimes(\sum_{i=1}^{n}t_ia_im_i)=\frac{1}{s}\otimes m'$ where $m'=\sum_{i=1}^{n}t_ia_im_i\in M$.
2. Considere the maps $f:S^{-1}R\otimes_R M\to S^{-1}M$, $f(\frac{a}{s}\otimes m)=\frac{am}{s}$ and $g:S^{-1}M\to S^{-1}R\otimes M$, $g(\frac{m}{s})=\frac{1}{s}\otimes m$. The maps f and g are well defined and $f\circ g=\mathrm{Id}_{S^{-1}M}$ and $g\circ f=\mathrm{Id}_{S^{-1}R\otimes_R M}$.

Exercice 4

Let R be a ring and $f\in R$ be a non nilpotent element. Let $\varphi:R[X]\to R_f$ the map defined by $\varphi(P)=P(\frac{1}{f})$.

1. Justifies that φ is a morphism of rings.
2. Show that φ is surjective.
3. Prove that $(fX-1)\subseteq\ker\varphi$.
4. Prove that $\ker\varphi=(fX-1)$.
5. Deduce that R_f is isomorphic to $\dfrac{R[X]}{(fX-1)}$.

☞ :

1. We see that $\varphi(PQ)=(PQ)(\frac{1}{f})=P(\frac{1}{f})Q(\frac{1}{f})=\varphi(P)\varphi(Q)$. And $\varphi(P+Q)=(P+Q)(\frac{1}{f})=P(\frac{1}{f})+Q(\frac{1}{f})=\varphi(P)+\varphi(Q)$. Moreover $\varphi(1)=1$.
2. Let $\frac{a}{f^n}\in R_f$. Set $P=aX^n$, then $\varphi(P)=a(\frac{1}{f})^n=\frac{a}{f^n}$. Hence φ is surjective.
3. We see that $\varphi(fX-1)=f\frac{1}{f}-1=0$, hence $fX-1\in\ker\varphi$, thus $(fX-1)\subseteq\ker\varphi$.
4. Let $P=\sum_{k=0}^{n}a_kX^k\in\ker\varphi$, That is $\sum_{k=0}^{n}a_k(\frac{1}{f})^k=0$. In the ring $R_f[X]$, we have $P=P-P(\frac{1}{f})=\sum_{k=0}^{n}a_k(X^k-(\frac{1}{f})^k)=\sum_{k=1}^{n}a_k(X^k-(\frac{1}{f})^k)$. But for each $1\le k\le n$, we have $X^k-(\frac{1}{f})^k=(X-\frac{1}{f})Q_k$ where $Q_k=\sum_{i=0}^{k-1}(\frac{1}{f})^iX^{k-1-i}\in R_f[X]$. Therefore
$$P=(X-\frac{1}{f})(\sum_{k=1}^{n}a_kQ_k)=(X-\frac{1}{f})Q=(fX-1)\frac{Q}{f}$$
where $Q=\sum_{k=1}^{n}a_kQ_k\in R_f[X]$. By multiplying with a suitable power of f, we get $f^mP=(fX-1)D$ where $D\in R[X]$. Now, we see that $(fX)^m-1=(fX-1)V$ for some $V\in R[X]$, hence $(fX)^mP=(fX-1)X^mD$, so $(1+(fX-1)V)P=(fX-1)X^mD$, thus $P=(fX-1)(X^mD-VP)\in(fX-1)$. It follows that $\ker\varphi=(fX-1)$.
5. By the isomorphism theorem, $\mathrm{Im}\,\varphi=R_f$ is isomorphic to $\frac{R[X]}{\ker\varphi}=\frac{R[X]}{(fX-1)}$.